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Title: The Advantages of Continuous Measurements Over Pass/Fail Data

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The Advantages of Continuous Measurements Over Pass/Fail Data

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Abstract

This article provides a justification for continuous measurements being more informative than pass/fail data. Namely, more information is provided for the same sample size or a smaller sample size is required for the same information. In the article, inference of the conformance probability, the probability of measurements meeting specifications, is considered. It is shown that continuous measurements provide dramatic advantages, especially when the conformance probability is high.

Key Words: lower confidence bound, probability of conformance, upper and lower specification limits, sample size.

Introduction

Recently two engineers asked me for advice in consecutive meetings about two aspects of the same problem. One engineer wanted a justification for requiring continuous measurements to be recorded, such as dimensions, rather than whether the measurements met specifications or not, so-called pass/fail data. The other engineer asked whether a smaller sample size could be justified for continuous measurements as compared with that for pass/fail data. Both engineers were interested in estimating the conformance probability, the probability that a measurement meets specification, which is typically 0.90 or higher. This article describes such a justification.

Preliminaries

Suppose that a continuous measurement has specification limits (L, U) and that the measurement follows a normal distribution with mean μ and variance σ^2, i.e., $N(\mu, \sigma^2)$. Then the conformance probability can be evaluated as:

$$p = \Phi((U - \mu)/\sigma) - \Phi((L - \mu)/\sigma), \tag{1}$$

where $\Phi()$ is the normal cumulative distribution function. See Figures 1-3 which displays the normal density and the specification limits for conformance probabilities of 0.90, 0.99 and 0.999, respectively.

*** Figures 1-3 about here ***

Let $\mathbf{x} = x_1, \ldots, x_n$ be a sample of n continuous measurements which are summarized by their sample mean and sample variance, $\bar{x}$ and s^2, respectively. For pass/fail data, the continuous measurements are passed if they are within the specification limits and failed if not. Then the number of passed data Y follows a binomial distribution with probability of success p and sample size n, i.e., $Binomial(n, p)$.

Now the goal is to provide inference on p, the conformance probability. For the pass/fail data, an $r \times 100$ % lower confidence bound can be used whose formula is

$$1/(1 + ((n - Y + 1)F^{-1}(r, 2n - 2Y + 2, 2Y)/Y)),$$

where $F^{-1}(r, df_1, df_2)$ is the rth quantile of an F distribution with parameters df_1 and df_2. Values of r such as 0.50 and 0.90 provide an estimate and lower confidence bound on p, respectively.

For continuous measurements, since p in (1) is a function of (μ, σ^2), a Bayesian approach provides a natural solution. Once the posterior distribution for (μ, σ^2) is obtained, as described below, the posterior distribution of p can be obtained by sampling from the (μ, σ^2) posterior and evaluating p. For inference, 0.50 and 0.10 quantiles of the posterior distribution of p provide an estimate and lower confidence bound on p, respectively.

Further details of the Bayesian approach are as follows. Let $\boldsymbol{\theta}$ denote a vector of parameters such as (μ, σ^2). The Bayesian approach combines prior information about $\boldsymbol{\theta}$ with the information about $\boldsymbol{\theta}$ contained in the data. The prior information is described by a probability density $\pi(\boldsymbol{\theta})$ known as the prior density and the information provided by the data is captured by the data sampling model $f(\mathbf{x}|\boldsymbol{\theta})$ known as the likelihood. The combined information is then described by another probability density $\pi(\boldsymbol{\theta}|\mathbf{x})$ called the posterior density. Bayes' Theorem (Degroot (1)) provides the way to calculate the posterior density, namely,

$$\pi(\boldsymbol{\theta}|\mathbf{x}) \propto f(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}).$$

For the prior density, I used the normal-inverted gamma (N-IG) density, i.e.,

$$\begin{aligned} &\sigma^2 \sim IG(c,d) \text{ or } 1/\sigma^2 \sim Gamma(c,d), \\ &\mu|\sigma^2 \sim N(a, b\sigma^2). \end{aligned}$$

Since the continuous measurements are independent and normally distributed, the data sampling model or likelihood is the product of n $N(\mu,\sigma^2)$ densities evaluated at the values $x_1,\ldots,x_n$. The N-IG prior is conjugate which means that the posterior has the same form, namely,

$$\begin{aligned} &\sigma^2 \sim IG(c_P, d_P), \\ &\mu|\sigma^2 \sim N(a_P, b_P\sigma^2), \end{aligned} \tag{2}$$

where

$$\begin{aligned} a_P &= ((a/b) + n\bar{x})/((1/b) + n) \\ b_P &= 1/((1/b) + n) \\ c_P &= c + n/2 \\ d_P &= d + (n-1)s^2/2 + (n/b)(\bar{x} - a)^2/(2((1/b) + n)). \end{aligned}$$

The form of the posterior also indicates how to sample from it; first draw σ^2 by taking the reciprocal of a Gamma random variable; then draw μ by generating a normal random variable.

A Comparison

A study was performed for various conformance probabilities p (0.90, 0.95, 0.99, 0.999, 0.9999, 0.99999) and sample sizes n (5, 10, 15, 20, 25, 50, 75, 100). For each case, 1000 sets of continuous measurement samples were simulated and then their associated pass/fail data were generated. For each sample, the 50% and 90% lower confidence bounds for p were calculated as described above for the pass/fail data and continuous measurements. The following prior density parameters were used: a=0.0, b=10000, c=0.01, d=0.01. The resulting density represents prior information that is very vague so that the Bayesian results reflect the information in the continuous measurements alone.

The results are presented in Tables 1-4 which are the average 50% and 90% lower confidence bounds (over the 1000 sets). The results demonstrate a compelling reason for using continuous measurements where possible. The continuous measurement estimates (50% lower confidence bounds) are closer to the true p's. Also, the continuous measurement 90% lower confidence bounds which reflect the information in the are much closer to the true p's; this is especially true for conformance probabilities near 1.0.

Returning to the engineers' questions which motivated this article, Tables 1-4 provide the following answers:

- Continuous measurements are more informative than pass/fail data. For a sample size n of 20 and conformance probability p of 0.90, the average 90% lower confidence bound for pass/fail data is 0.76205 as compared with 0.80347 for continuous measurements. A more dramatic example is for a sample size n of 25 and conformance probability p of 0.999 in which the average 90% lower confidence bound for pass/fail data is 0.91077 as compared with 0.98963 for continuous measurements.

- A smaller sample size for continuous measurements provides the same information as that for pass/fail data with a larger sample size. For a conformance probability p of 0.90, the average 90% lower confidence bound for pass/fail data based on a sample size n of 20 is 0.76205 as compared with 0.78580 (0.02 better) for continuous measurements based on a sample size n of 15. A more dramatic example is for a conformance probability p of 0.999 in which the average 90% lower confidence bound for pass/fail data based on a sample size n of 75 is 0.96822 as compared with 0.96729 for continuous measurements based on a sample size n of 10!

Table 1: Average 50% Lower Bound Using Pass/Fail Data

	sample size							
p	5	10	15	20	25	50	75	100
0.90	0.87055	0.93303	0.95484	0.86853	0.89447	0.86750	0.92473	0.85381
0.95	0.87055	0.93303	0.95484	0.96594	0.93377	0.94688	0.95126	0.94349
0.99	0.87055	0.93303	0.95484	0.96594	0.89447	0.98623	0.99080	0.98327
0.999	0.87055	0.93303	0.95484	0.96594	0.97265	0.98623	0.99080	0.99309
0.9999	0.87055	0.93303	0.95484	0.96594	0.97265	0.98623	0.99080	0.99309
0.99999	0.87055	0.93303	0.95484	0.96594	0.97265	0.98623	0.99080	0.99309

Table 2: Average 50% Lower Bound Using Continuous Measurements

	sample size							
p	5	10	15	20	25	50	75	100
0.90	0.86595	0.88245	0.88818	0.88984	0.89130	0.89600	0.89620	0.89607
0.95	0.91906	0.92993	0.93808	0.93931	0.94057	0.94571	0.94656	0.94707
0.99	0.97033	0.97828	0.98277	0.98487	0.98528	0.98770	0.98840	0.98870
0.999	0.99123	0.99491	0.99630	0.99706	0.99771	0.99836	0.99862	0.99870
0.9999	0.99741	0.99870	0.99923	0.99952	0.99958	0.99976	0.99979	0.99983
0.99999	0.99876	0.99957	0.99978	0.99988	0.99992	0.99996	0.99997	0.99998

Table 3: Average 90% Lower Bound Using Pass/Fail Data

p	sample size 5	10	15	20	25	50	75	100
0.90	0.52663	0.67239	0.73086	0.76205	0.77898	0.82497	0.84033	0.84944
0.95	0.58013	0.72625	0.79137	0.82132	0.84050	0.88715	0.90075	0.90936
0.99	0.62009	0.78220	0.84466	0.87911	0.89669	0.94043	0.95469	0.96294
0.999	0.63053	0.79315	0.85564	0.88975	0.91077	0.95359	0.96822	0.97588
0.9999	0.63096	0.79393	0.85770	0.89111	0.91189	0.95493	0.96962	0.97711
0.99999	0.63096	0.79420	0.85770	0.89118	0.91201	0.95499	0.96977	0.97724

Table 4: Average 90% Lower Bound Using Continuous Measurements

p	sample size 5	10	15	20	25	50	75	100
0.90	0.66712	0.75178	0.78580	0.80347	0.81550	0.84543	0.85588	0.86162
0.95	0.74909	0.82285	0.85645	0.87220	0.88208	0.90849	0.91734	0.92230
0.99	0.85780	0.91623	0.94082	0.95281	0.95798	0.97219	0.97690	0.97922
0.999	0.92724	0.96720	0.98010	0.98559	0.98963	0.99450	0.99609	0.99669
0.9999	0.96328	0.98682	0.99352	0.99609	0.99704	0.99882	0.99920	0.99943
0.99999	0.97550	0.99375	0.99730	0.99859	0.99912	0.99973	0.99984	0.99990

Conclusions

Intuitively, continuous measurements are more informative than pass/fail data. This article provides a justification. Namely, more information is provided for the same sample size or a smaller sample size is required for the same information. There are dramatic advantages when the conformance probability is high. Consequently, continuous measurements are recommended where possible. Often, this may be as simple as requiring that continuous measurements be recorded rather than checking a box that the measurements met specifications.

Acknowledgements

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References

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Figure 2: 0.99 Conformance probability

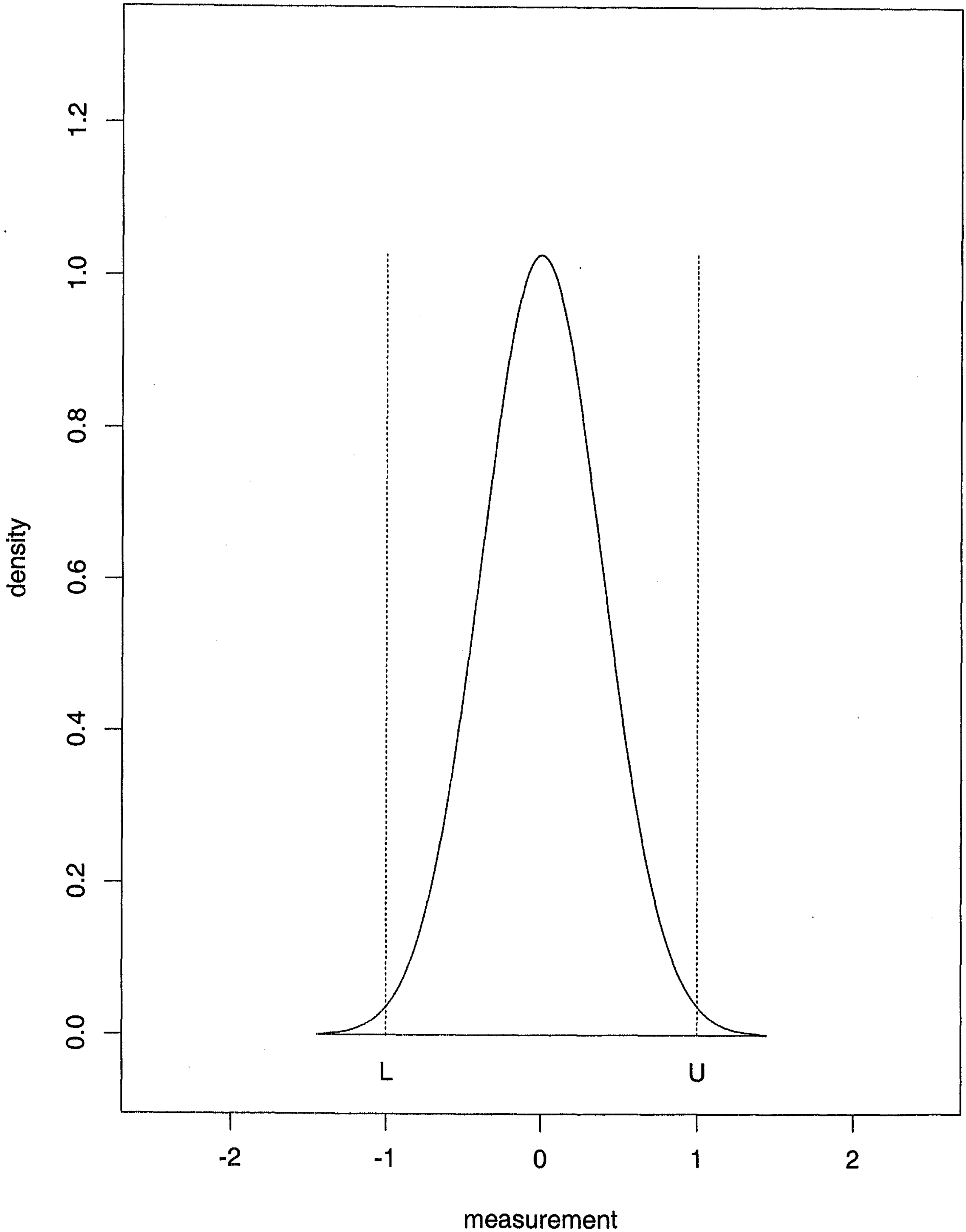

Figure 1: 0.90 conformance probability

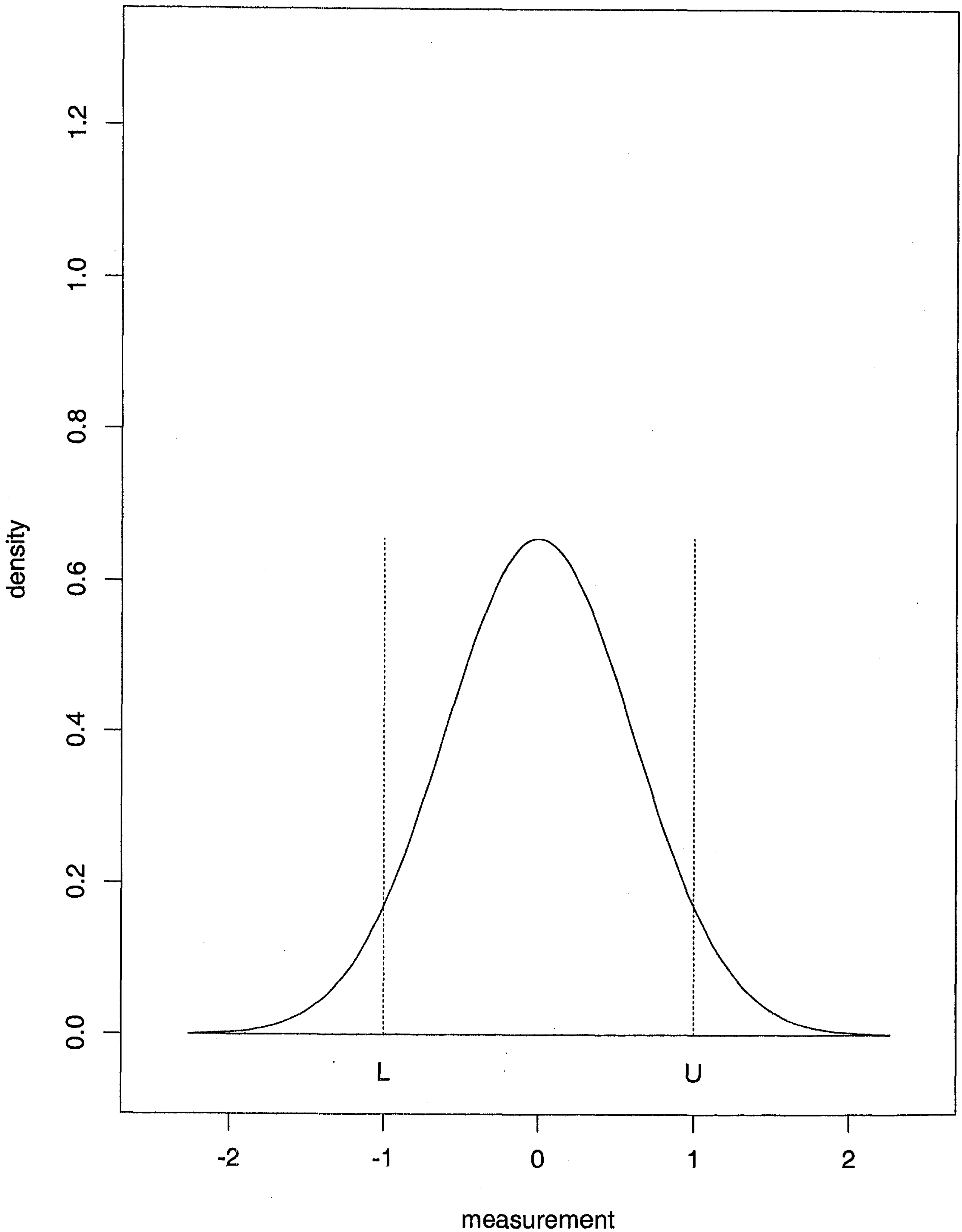

Figure 3: 0.999 Conformance Probability